INTERMEDIATE ENTROPIES FOR RATIONAL MAPS

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ABSTRACT. We prove that any rational map verifies the intermediate entropy property.

1. INTRODUCTION

In 1980 Katok proved an outstanding result for $C^{1+\alpha}$ diffeomorphisms on compact surfaces: any ergodic measure with positive entropy is hyperbolic. In particular, given a $C^{1+\alpha}$ surface diffeomorphism $f : S \to S$, for any $h \in [0, h_{top}(f))$ there exists an ergodic f-invariant probability measure μ such that $h_\mu(f) = h$. This last property is called the *intermediate entropy property.* Katok conjectured that the intermediate entropy property holds for any smooth system. Since then, many partial results have appeared in the literature (see for instance [QS], [Su1], [GSW] and [Su2]).

In this note[1] we will work in other direction. Since for the best of our knowledge there is no known result for smooth endomorphisms, we address this question. More specifically, we will consider rational endomorphism of the Riemann sphere.

Theorem 1.1. *Any rational map of the Riemann sphere verifies the intermediate entropy property.*

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2. PRELIMINARIES

Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$ and let $\mathcal{M}_f$ be the convex space of f-invariant probability measures on $\hat{\mathbb{C}}$. Recall that this set is compact when it is provided with the weak* topology. In particular, for a continuous function $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$, mostly called potential in this work, the map

$$\mu \mapsto \int \varphi d\mu$$

is continuous.

The topological entropy of f is, roughly speaking, the uniform exponential growth rate of the topological complexity given by the iteration of f. It also coincides -via the variational principle- with the supremum of all the measure-theoretic entropies with respect to measures in $\mathcal{M}_f$. Combining inequalities obtained by Gromov [G], Misiurewicz and Przytycki [MP], the topological entropy was proved to be equal to $\log(d)$. Later, from the ergodic point of view, Ljubich [L] proved the

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existence and uniqueness of a measure maximizing the entropy (see also [FLM]) together with some relevant ergodic properties of that measure.

A generalization of the topological entropy is the *topological pressure*. It consists in associating a weight to each point of $\hat{\mathbb{C}}$ in order to measure the complexity of the system in a non-uniformly way. The weight of a point $z \in \hat{\mathbb{C}}$ is given by the value $\varphi(z)$, where $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ is a potential. This rough idea of weighted entropy can be defined through the variational principle.

Definition 2.1 (Variational Principle)**.** *Let $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ be a continuous potential. The* topological pressure $P(\varphi)$ *of φ is defined as*

$$P(\varphi) = \sup_{\nu \in \mathcal{M}_f} \left\{ h_\nu(f) + \int \varphi d\nu \right\},$$

where $h_\nu(f)$ denotes the measure-theoretic entropy of f. If $m \in \mathcal{M}_f$ maximizes the supremum, we will call it equilibrium measure *for φ.*

Depending on the dynamical system, and certainly on the regularity of the potential, many answers can be found around the existence and uniqueness of equilibrium measures. In our setting it is well known that any continuous potential admits an equilibrium measure. In fact, this directly follows from the upper-semicontinuity regularity of the entropy map $\mu \to h_\mu(f)$ and the variational principle (see [L]). In terms of the uniqueness of such a measure, additional assumptions are needed in general. The easier case occurs when the potential is $\varphi \equiv 0$. In this case there is a unique equilibrium measure associated to it: the measure of maximal entropy.

In order to ensure the uniqueness of an equilibrium measure for an arbitrary potential, we will use the following fundamental result due to Denker and Urbański [DU].

Theorem 2.2. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$ and $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ a Hölder-continuous potential verifying*

$$\sup_{z \in \hat{\mathbb{C}}} \varphi(z) < P(\varphi). \tag{1}$$

Then φ admits a unique equilibrium measure m_φ.

Remark 2.3. *By convexity of the map*

$$\mu \mapsto h_\mu(f) + \int \varphi d\mu,$$

if an equilibrium measure is unique, then it is necessarily ergodic.

The strategy of the proof of Theorem 1.1 is to get the desired ergodic measure as an equilibrium measure. Hence, to establish whether or not (1) is satisfied will be crucial. With this goal in mind, the proposition below is outstanding. It roughly speaking states that repelling periodic orbits do not capture the weighted chaos of the system (see Proposition 4.1 in [IRRL]). A remarkable aspect of this result is that this property holds despite the fact that repelling periodic points are dense in the Julia set (see [J] and [F]), whereas at the same time, the Julia set concentrates the topological chaos of the dynamical system.

Before giving the proper statement of the proposition, we recall that a periodic point $z_0 \in \hat{\mathbb{C}}$ with period $n \geqslant 1$ is *repelling* if $|(f^n)'(z_0)| > 1$.

Proposition 2.4. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational map of degree $d \geqslant 2$ and $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ a Hölder-continuous potential. Then, for each integer $n \geqslant 1$ and each repelling periodic point z_0 of f with period n, we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}\varphi(f^k z_0) < P(\varphi).$$

3. Proof of the main theorem

Let $\mathcal{O}$ be any repelling periodic orbit of f with period n. We define $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ as

$$\varphi(z) = \frac{1}{1 + d(z, \mathcal{O})},$$

where d is any distance inducing the standard topology of $\hat{\mathbb{C}}$. Observe that φ is a Hölder-continuous potential and it attains its maximum exactly on $\mathcal{O}$. In particular, any measure maximizing the map $\mu \mapsto \int \varphi d\mu$ coincides with the measure $\nu \in \mathcal{M}_f$ supported on the periodic orbit $\mathcal{O}$.

Let $z_0 \in \mathcal{O}$. Then, by Proposition 2.4, for every $t > 0$, we have

$$\sup_{z\in\hat{\mathbb{C}}}(t\varphi)(z) = t = \frac{1}{n}\sum_{k=0}^{n-1}(t\varphi)(f^k z_0) < P(t\varphi).$$

We can then apply Theorem 2.2 to ensure that the potential $t\varphi$ admits a unique equilibrium measure $m_{t\varphi}$ for every $t > 0$, which is ergodic by Remark 2.3.

Before continuing the proof of the theorem, let us recall a final and standard property of the pressure map (see Chapter 4 in [Ke] for details).

Theorem 3.1. *Let $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ be as before. Then, the map $t \mapsto P(t\varphi)$ is convex and strictly increasing on $(0, \infty)$. Moreover, it is differentiable with derivative*

$$\frac{d}{dt}P(t\varphi) = \int \varphi dm_{t\varphi}.$$

From the upper-semicontinuity property of the entropy map and the continuity of the map $\mu \mapsto \int \varphi d\mu$ it is not hard to prove the following.

Corollary 3.2. *Let $\varphi : \hat{\mathbb{C}} \to \mathbb{R}$ be as before. Then, the map $t \mapsto \int \varphi dm_{t\varphi}$ is continuous on $(0, \infty)$. In particular, the pressure map $t \mapsto P(t\varphi)$ is C^1.*

Many conclusions follow after the theorem and corollary above. We leave the details to the reader.

Fact 1. Any weak* limit ν' of $(m_{t\varphi})_t$ as $t \to \infty$ verifies

$$\lim_{t\to\infty}\int \varphi dm_{t\varphi} = \int \varphi d\nu'.$$

Fact 2. Any weak* limit ν' of $(m_{t\varphi})_t$ as $t \to \infty$ must be a maximizing measure for φ, so the sequence $(m_{t\varphi})_t$ converge to ν.

Fact 3. By continuity of the maps $t \mapsto P(t\varphi)$ and $t \mapsto \int \varphi dm_{t\varphi}$, we conclude that the map $t \mapsto h_{m_{t\varphi}}(f)$ is continuous on $(0, \infty)$.

Finally, again by the upper-semicontinuity property of the entropy map, we know that

$$\limsup_{t\to\infty} h_{m_{t\varphi}}(f) \leqslant h_\nu(f) = 0,$$

so the map $t \to h_{m_{t\varphi}}(f)$ ranges continuously from $\log(d)$ at $t = 0$, to 0 at $t = \infty$. Therefore, for every $h \in (0, \log(d))$ there exists $t_h \in (0, \infty)$ such that $h_{m_{t_h\varphi}}(f) = h$. Since $m_{t_h\varphi}$ is ergodic, this concludes the proof of the theorem.

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