ASAI CUBE L-FUNCTIONS AND THE LOCAL LANGLANDS CONJECTURE

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Abstract. We work over non-Archimedean local fields F of characteristic 0 as well as characteristic p. Let E/F be a separable cubic extension and let $\mathbf{H}$ be an ambient group of type D_4, which has triality corresponding to E. We choose a maximal Levi subgroup $\mathbf{M}$ isogenous to $G = \mathrm{GL}_2(E)$. The Langlands-Shahidi method applied to $(\mathbf{H}, \mathbf{M})$ attaches an Asai cube γ-factor to an irreducible smooth generic representation π of G. If σ is the Weil-Deligne representation corresponding to π via local Langlands, we prove that the Asai cube γ-factor of π is the γ-factor of the Weil-Deligne representation obtained from σ via cubic tensor induction from E to F. A consequence is that Asai cube γ-factors become stable under twists by highly ramified characters.

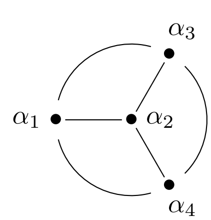
Introduction

Let $\mathbf{G}$ be a quasi-split connected reductive group defined over a non-Archimedean local field F and let r be a complex representation of the Langlands dual group ${}^LG$. The Langlands-Shahidi method produces γ-factors $\gamma(s, \pi, r, \psi)$ in many situations for r, a generic representation π and a smooth character $\psi : F \to \mathbb{C}^\times$. We let $\mathcal{W}_F$ denote the absolute Weil group of F, and $\mathcal{W}'_F$ the Weil-Deligne group. The local Langlands correspondence –still mostly conjectural–, attaches to π a Langlands parameter $\phi_\pi : \mathcal{W}'_F \to {}^LG$ (up to ${}^LG^\circ$-conjugacy). Whenever such a correspondence is established, we should have the following main equality

$$\gamma(s, \pi, r, \psi) = \gamma(s, r \circ \phi_\pi, \psi). \tag{1}$$

On the right hand side, they are the Galois γ-factors of Deligne and Langlands.

In this paper, we focus on a case which has not been looked at seriously before. Our group $\mathbf{G}$ is $\mathrm{Res}_{E/F}\mathrm{GL}_2$, restriction of scalars, where E is a separable cubic extension of F. In the Langlands-Shahidi method (see [26] and [23]) we choose the ambient quasi-split group $\mathbf{H}$ over F to be of type D_4, with triality corresponding to E/F. We choose $\mathbf{H}$ to be simply connected, so that ${}^LH$ is of adjoint type. We let $\mathbf{M}$ be the maximal Levi subgroup of $\mathbf{H}$ obtained by removing the central root α_2 from the diagram:



The local Langlands correspondence for GL_2 is known, see [17] and the more detailed account of [2]. Let π be a (generic) smooth irreducible representation of $\mathrm{GL}_2(E)$, and σ the corresponding degree 2 representation of $\mathcal{W}'_E$ obtained via the local Langlands correspondence. On the one hand, we have that the Langlands-Shahidi method studies the pair $(\mathbf{H}, \mathbf{M})$ and the adjoint action of the Langlands dual group ${}^LM$ on the Lie algebra of the unipotent radical of the parabolic of ${}^LH$ with Levi ${}^LM$. This produces an Asai cube representation $r_{\mathcal{A}}$ of ${}^LM$; hence, by composition, a degree 8 representation ${}^{\otimes}\mathrm{I}$ of ${}^LG$. On the other side of the Langlands correspondence, we let ${}^{\otimes}\mathrm{I}(\sigma)$ be the representation of $\mathcal{W}'_F$ obtained from σ by tensor induction from E to F. We prove the following main equality

$$\gamma(s, \pi, {}^{\otimes}\mathrm{I}, \psi) = \gamma(s, {}^{\otimes}\mathrm{I}(\sigma), \psi), \tag{2}$$

between Langlands-Shahidi γ-factors and the Galois γ-factors of Deligne and Langlands. We further have the following equalities

$$\begin{aligned} L(s, \pi, {}^{\otimes}\mathrm{I}) &= L(s, {}^{\otimes}\mathrm{I}(\sigma)) \\ \varepsilon(s, \pi, {}^{\otimes}\mathrm{I}, \psi) &= \varepsilon(s, {}^{\otimes}\mathrm{I}(\sigma), \psi) \end{aligned} \tag{3}$$

which we derive from the main equation for γ-factors.

When F has positive characteristic, we used a local-to-global argument in [12, 14] together with a characterization of γ-factors in order to prove the corresponding version of equation (2) for exterior square, symmetric square, and regular Asai γ-factors. The local-to-global result of [13] was generalized over function fields in [8], and our previous results on (2) are a consequence of the general considerations of Gan-Lomelí in characterisit p. We observe that equalities (2) and (3) are true for unramified principal series over any non-Archimedean F [16]; and, in the Archimedean case [24].

When F has characteristic 0, we also use a local-to-global argument and a characterization of γ-factors $\gamma(s, \pi, r_{\mathcal{A}}, \psi)$ through a list of axioms. All but one of the axioms are local, and the corresponding axioms for $\gamma(s, {}^{\otimes}\mathrm{I}(\sigma), \psi)$ on the Galois side are easy to prove. The global functional equation is the remaining property expressing the local-to-global compatibility, it involves partial L-functions and γ-factors. We take advantage that the strong Artin conjecture is known for 2-dimensional Σ with solvable image [18, 28]. For $\gamma(s, {}^{\otimes}\mathrm{I}(\sigma), \psi)$, such a functional equation is available since σ appears as a component of a global representation Σ for which the L-function of $r \circ \Sigma$ has a functional equation. The Langlands-Shahidi method supplies the functional equation on the corresponding automorphic L-functions side. The representations π and σ are globalized in a convenient way, and they are components of global representaitons Π and Σ at a place v_0 of a global field k. We then prove equality (2) at all places except, perhaps, the place v_0. Finally, we compare both global functional equations to get our result.

It is a consequence of our method that γ-factors satisfy a stability property: if we twist π by a sufficiently ramified character χ then $\gamma(s, \pi \otimes \chi, r, \psi)$ depends only on the central character π and not on π itself. See [8] for a general statement in the case of characteristic p.

1. Asai cube L-functions

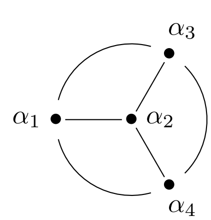
Let E be a cubic extension of F in $\bar{F}$; we view $\mathcal{W}_E = \mathcal{W}(\bar{F}/E)$ as an open subgroup of $\mathcal{W}_F$, of index 3, and similarly for the Weil-Deligne groups $\mathcal{W}'_E$ and $\mathcal{W}'_F$.

In this section, we want to use the Langlands-Shahidi method to obtain the automorphic γ-factors for $\mathrm{GL}_2(E)$ corresponding to the previous tensor induction process. More precisely, we see $\mathrm{GL}_2(E)$ as $\mathbf{G}(F)$, where $\mathbf{G} = \mathrm{Res}_{E/F}\,\mathrm{GL}_2$. The L-group of $\mathbf{G}$ is the semidirect product ${}^LG = \mathrm{GL}_2^J \rtimes \mathcal{W}_F$, where $J = \mathrm{Hom}_F(E, \bar{F})$, and $\mathcal{W}_F$ acts on $\mathrm{GL}_2(\mathbb{C})^J$ via its natural action on J. The group ${}^LG$ has a natural 8-dimensional representation $\mathrm{I}^{\otimes}$: if j is the given embedding of E into $\bar{F}$, $\mathrm{GL}_2(\mathbb{C})^J$ acts on $\mathbb{C}^2$ via its j-component, and $\mathcal{W}_E$ acts on $\mathbb{C}^2$ trivially; tensor induction to ${}^LG$ then gives $\mathrm{I}^{\otimes}$.

We now construct a quasi-split group $\mathbf{H}$ over F, and a maximal parabolic subgroup $\mathbf{P}$ of $\mathbf{H}$, with a Levi subgroup $\mathbf{M}$ close enough to $\mathbf{G}$ that the action of ${}^LM$ on the Lie algebra ${}^L\mathfrak{n}$ of the unipotent radical of ${}^LP$ gives rise to the representation $\mathrm{I}^{\otimes}$ of ${}^LG$. It is not necessary for us to describe $\mathbf{H}$ explicitly, as only the L-group picture will be useful to us. Because $\mathbf{H}$ is quasi-split, its indexed root datum [27] § 16.2.1 specifies $\mathbf{H}$ up to isomorphism, and we work with the dual root datum $(X, \Phi, X^\vee, \Phi^\vee)$, where X is the group of characters of a maximal torus in $\widehat{H}$ (the dual group of $\mathbf{H}$) and Φ is the set of roots of $\widehat{T}$ in $\widehat{H}$; the group $X^\vee$ is the group of cocharacters of $\widehat{T}$ and $\Phi^\vee$ is the set of coroots, equipped with a bijection $\alpha \mapsto \alpha^\vee$ of Φ onto $\Phi^\vee$; finally the group $\Gamma_F = \mathrm{Gal}(\bar{F}/F)$ acts linearly on X (via a finite quotient), and that action preserves Φ, the dual action on $X^\vee$ preserving $\Phi^\vee$, compatibly with the bijection $\alpha \mapsto \alpha^\vee$.

Our group ${}^LH$ has type D_4 and to describe its root datum we use Bourbaki notaion [1]. However, we prefer to separate the roles of X and $X^\vee$ (and not identify them via some Killing form), so that we let X be the set of vectors in $V = \mathbb{R}^4$ with integer coordinates adding to an even number; we write $(e_1, \dots, e_4)$ for the canonical basis of V, and choose $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2 - e_3$, $\alpha_3 = e_3 - e_4$ and $\alpha_4 = e_3 + e_4$ as a basis of Φ. The vector space $V^\vee$ dual to V has the basis $(e_1^\vee, \dots, e_4^\vee)$ dual to $(e_1, \dots, e_4)$; the simple coroots are $\alpha_1^\vee = e_1^\vee - e_2^\vee$, $\alpha_2^\vee = e_2^\vee - e_3^\vee$, $\alpha_3^\vee = e_3^\vee - e_4^\vee$ and $\alpha_4^\vee = e_3^\vee + e_4^\vee$; the lattice $X^\vee$ in $V^\vee$ is generated by $\frac{1}{2}(e_1^\vee + e_2^\vee + e_3^\vee + e_4^\vee)$ and $e_2^\vee$, $e_3^\vee$, $e_4^\vee$.

Writing $\mathfrak{S}$ for the group of permutations of $\{1, 3, 4\}$, we have a natural action of $\mathfrak{S}$ on V preserving X, fixing α_2 and permuting α_1, α_3, α_4 according to the indices.



Any group homomorphism $\Gamma_F \to \mathfrak{S}$ then gives a group root datum for ${}^LH$, where $\mathbf{H}$ is a quasi-split group over F. For example, with E a cubic extension of F in $\bar{F}$ as at the beginning, an identification of $\mathrm{Hom}_F(E, \bar{F})$ with $\{1, 3, 4\}$ gives a homomorphism $\Gamma_F \to \mathfrak{S}$ producing the group ${}^LH$ we seek.

Remark 1.1. The construction just described is valid with any field in place of F, and we shall use it for a global field k and a cubic separable extension, thus producing groups $\mathbf{H}/k$ and ${}^L H_k$. If v is a place of k, the root datum for the L-group ${}^L H_{k_v}$ of $\mathbf{H} \otimes_k k_v$ is obtained from the root datumf of ${}^L H_k$ by composing the action of Γ_k with the homomorphism $\Gamma_{k_v} \to \Gamma_k$ coming from the completion (such a homomorphism depends on an isomorphism of $\bar{k}$ with the algebraic closure of k in $\bar{k}_v$, but changing it changes ${}^L H_{k_v}$ to an isomorphic group). Note that even if the homomorphism $\Gamma_k \to \mathfrak{S}$ is surjective, the local homomorphism $\Gamma_{k_v} \to \mathfrak{S}$ might not be. For example, at a split place v a global cubic extension l/k is $l_v \simeq k_v \times k_v \times k_v$, the local homomorphism $\Gamma_{k_v} \to \mathfrak{S}$ is trivial.

We use the maximal parabolic subgroup ${}^L P$ of ${}^L H$ which corresponds to omitting the simple root α_2; if $\widehat{M}$ is the Levi subgroup of $\widehat{\Phi}$ containing $\widehat{T}$, the roots of $\widehat{T}$ in $\widehat{M}$ are the roots in Φ which are linear combinations of α_1, α_3, α_4, whereas the roots of $\widehat{T}$ in ${}^L\mathfrak{n}$ are the (positive) roots in Φ where α_2 appears with a positive coefficient. The adjoint representation of $\widehat{M}$ on ${}^L\mathfrak{n}$ has two irreducible components r_i, $i = 1, 2$, and the corresponding roots of $\widehat{T}$ are the roots in Φ where α_2 appears with coefficient i – we use $r = r_1$.

Now we relate ${}^L G$ and ${}^L M$, and r with $\mathrm{I}^{\otimes}$. With the chosen identification of J with $\{1,3,4\}$, ${}^L G$ becomes $\mathrm{GL}_2(\mathbb{C})^{\{1,3,4\}} \rtimes \mathcal{W}_F$; let us describe the corresponding root datum and relate it to the root datum for ${}^L M$. For a root datum $(Y, \Psi, Y^\vee, \Psi^\vee)$ for ${}^L G$, we can take

$$Y = Y_1 \oplus Y_2 \oplus Y_3 \text{ with } Y_i = \mathbb{Z}e_i \oplus \mathbb{Z}f_i,$$

where $\alpha_i = e_i - f_i$ for $i = 1, 3, 4$ as simple roots; similarly

$$Y^\vee = Y_1^\vee \oplus Y_2^\vee \oplus Y_3^\vee \text{ with } Y_i^\vee = \mathbb{Z}e_i^\vee \oplus \mathbb{Z}f_i^\vee,$$

where $\alpha_i^\vee = e_i^\vee - f_i^\vee$ for $i = 1, 3, 4$ – the duality is the obvious one, and Γ_F acts via its action on $\{1, 3, 4\}$.

Consider the quotient ${}^L\overline{G}$ of ${}^L G$ by the central subgroup made out of central elements (x_1, x_3, x_4) in $\mathrm{GL}_2(\mathbb{C})^{\{1,3,4\}}$ such that $x_1 x_3 x_4 = 1$; the corresponding root datum is $(Z, \Psi, \bar{Z}^\vee, \overline{\Psi}^\vee)$, where Z is the sublattice of Y of elements $a_1 e_1 + b_1 f_1 + a_3 e_3 + b_3 d_3 + a_4 e_4 + b_4 f_4$ such that $a_1 + b_1 = a_3 + b_3 = a_4 + b_4$; then $\bar{Z}^\vee$ is the corresponding quotient of $Y^\vee$ and $\overline{\Psi}^\vee$ is the image of $\Psi^\vee$ in $\bar{Z}^\vee$. The action of Γ_F is, again, obtained via the action on $\{1, 3, 4\}$. One verifies immediately that the root datum $(Z, \Psi, \bar{Z}^\vee, \overline{\Psi})$ is isomorphic to the one for ${}^L M$ by sending α_i to α_i, for $i = 1, 3, 4$, and $f_1 + f_2 + f_3$ to α_2, so that dually $\bar{\alpha}_i^\vee$ in $\bar{Z}^\vee$ is sent to $\alpha_i^\vee$, for $i = 1, 3, 4$, wheras the image of $e_1^\vee + f_1^\vee$ in $\bar{Z}^\vee$ (which is the same as the image of $e_2^\vee + f_2^\vee$ or $e_3^\vee + f_3^\vee$), is sent to $\alpha_1^\vee + \alpha_3^\vee + \alpha_4^\vee - 2\alpha_2^\vee$ (so that $e_1^\vee$ is sent to $\alpha_1^\vee - \alpha_2^\vee + \frac{1}{2}(\alpha_3^\vee + \alpha_4^\vee)$).

It follows that we can choose an isomorphism φ of ${}^L\bar{G}$ onto ${}^L M$, compatible with the isomorphism of root data just described. Since the representation r of $\widehat{M}$ on ${}^L\mathfrak{n}$ as roots α_2, $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, $\alpha_2 + \alpha_4$, $\alpha_1 + \alpha_2 + \alpha_3$, $\alpha_1 + \alpha_2 + \alpha_4$, $\alpha_2 + \alpha_3 + \alpha_4$, $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$, we see that through $\widehat{G} \to \widehat{\overline{G}} \to \widehat{M}$ r gives rise to the tensor product representation of $\widehat{G} = \mathrm{GL}_2(\mathbb{C})^{\{1,3,4\}}$, $\{g_1, g_3, g_4\} \mapsto g_1 \otimes g_3 \otimes g_4$. That identification even extends to ${}^L G$ and its action via $\mathrm{I}^{\otimes}$: indeed in the representation of ${}^L M$ on ${}^L\mathfrak{n}$ we can choose bases for the root subspaces so that the action of $\mathfrak{S}$ (hence Γ_F) on these vectors is given by its action on the roots via the representation of $\{1, 3, 4\}$. It is then clear that via ${}^L G \to {}^L M$, r does indeed give $\mathrm{I}^{\otimes}$.

2. LS method and Asai cube

Let $\mathscr{A}$ denote the class of triples $(E/F, \pi, \psi)$ consisting of: a degree 3 étale algebra E over a non-Archimedean local field F; a smooth irreducible (complex) representation π of $\mathrm{GL}_2(E)$; a non-trivial character $\psi : F \to \mathbb{C}^\times$.

The Langlands-Shahidi method produces γ-factors

$$\gamma(s, \pi, \mathrm{I}^{\otimes}, \psi) \text{ for } (E/F, \pi, \psi) \in \mathscr{A}.$$

More precisely, the LS method attaches a factor $\gamma(s, \pi', r_{\mathcal{A}}, \psi)$ to a representation π' of M; which we identify as a subgroup of G

$$M \hookrightarrow \left\{ g \in G \,|\, \mathrm{N}_{E/F}(\det g) = 1 \right\}.$$

The factor $\gamma(s, \pi, \mathrm{I}^{\otimes}, \psi)$ is by definition the Langlands-Shahidi factor $\gamma(s, \pi', r_{\mathcal{A}}, \psi)$, where π' is the (unique) ψ-generic component of the restricition of π to M.

We also have the global class $\mathscr{A}^{\mathrm{glob}}$ of quadruples $(K/k, \Pi, \Psi, S)$ consisting of: a separable cubic extension of global fields K/k; a (unitary) globally generic representation $\Pi = \otimes' \Pi_v$ of $\mathrm{GL}_n(\mathbb{A}_K)$; a non-trivial character $\Psi : k \backslash \mathbb{A}_k \to \mathbb{C}^\times$; and S a finite set of places of K containing all Archimedean v and such that Π_v is unramified for $v \notin S$.

2.1. **Characterization of γ-factors.** Let us recall the main theorem of the Langlands-Shahidi method L-functions, γ-factors and root numbers. Namely, Theorem 3.5 of [26] and Theorem 4.1 of [23]. There exists a system of rational γ-factors on $\mathscr{A}$. They are uniquely determined by the following properties:

(i) *(Naturality).* Let $(E/F, \pi, \psi) \in \mathscr{A}$. Let $\eta : E'/F' \to E/F$ be an isomorphism of degree 3 extensions over the non-Archimedean local fields F and F'. Let $(E'/F', \pi', \psi') \in \mathscr{A}$ be the triple obtained via η. Then

$$\gamma(s, \pi, {}^{\otimes}\mathrm{I}, \psi) = \gamma(s, \pi', {}^{\otimes}\mathrm{I}, \psi').$$

(ii) *(Isomorphism).* Let $(E/F, \pi_j, \psi) \in \mathscr{A}$, $j = 1, 2$. If $\pi_1 \cong \pi_2$, then

$$\gamma(s, \pi_1, {}^{\otimes}\mathrm{I}, \psi) = \gamma(s, \pi_2, {}^{\otimes}\mathrm{I}, \psi).$$

(iii) *(Compatibility with Artin factors).* Let $(E/F, \pi, \psi) \in \mathscr{A}$ be such that π has an Iwahori fixed vector. Also, π can be a generic representation of $\mathbf{G}(\mathbb{R})$. Let $\phi_\pi : \mathcal{W}_F' \to {}^L M$ be the Langlands parameter corresponding to π. Then

$$\gamma(s, \pi, {}^{\otimes}\mathrm{I}, \psi) = \gamma(s, {}^{\otimes}\mathrm{I}(\phi_\pi), \psi).$$

(iv) *(Multiplicativity).* Let $(E/F, \pi, \psi) \in \mathscr{A}$ be such that

$$\pi \hookrightarrow \mathrm{Ind}(\chi_1 \otimes \chi_2),$$

the generic constituent obtained from parabolic induction with χ_i, $i = 1, 2$, characters of $E^\times$. Identify χ_i with a character of $\mathcal{W}_E$ via class field theory. Then

$$\begin{aligned}\gamma(s, \pi, {}^{\otimes}\mathrm{I}, \psi) = &\gamma(s, \chi_1|_{F^\times}, \psi)\gamma(s, \chi_2|_{F^\times}, \psi) \\ &\gamma(s, \mathrm{I}_{E/F}(\chi_1\chi_2^{-1}) \otimes \chi_2|_{F^\times}, \psi)\gamma(s, \mathrm{I}_{E/F}(\chi_2\chi_1^{-1}) \otimes \chi_1|_{F^\times}, \psi),\end{aligned}$$

where $\mathrm{I}_{E/F}$ denotes induction from $\mathcal{W}_E$ to $\mathcal{W}_F$.

(v) *(Split places).* Let $(E/F, \pi, \psi) \in \mathscr{A}$ be such that E is the separable algebra $F \times F \times F$. Then π is given by $\pi_1 \otimes \pi_2 \otimes \pi_3$, where each π_j, $j = 1, \ldots, 3$, is a smooth generic representation of $\mathrm{GL}_2(F)$. For each j, let $\phi_{\pi_j} : \mathcal{W}'_F \to {}^L M$ be the corresponding Langlands parameter. Then

$$\gamma(s, \pi, {}^{\otimes}\mathrm{I}, \psi) = \gamma(s, \phi_{\pi_1} \otimes \phi_{\pi_2} \otimes \phi_{\pi_3}, \psi).$$

(vi) *(Dependence on ψ).* Let $(E/F, \pi, \psi) \in \mathscr{A}$. Given $a \in F^\times$, let $\psi^a : F \to \mathbb{C}^\times$ be the character given by $\psi^a(x) = \psi(ax)$. Then

$$\gamma(s, \pi, {}^{\otimes}\mathrm{I}, \psi^a) = \omega_\pi(a)^2 \, |a|_F^{4(s-\frac{1}{2})} \cdot \gamma(s, \pi, {}^{\otimes}\mathrm{I}, \psi).$$

(vii) *(Functional equation).* Let $(k, \Pi, \Psi, S) \in \mathscr{A}^{\mathrm{glob}}$. Then

$$L^S(s, \Pi, {}^{\otimes}\mathrm{I}) = \prod_{v \in S} \gamma(s, \Pi_v, {}^{\otimes}\mathrm{I}_v, \Psi_v) L^S(s, \widetilde{\Pi}, {}^{\otimes}\mathrm{I}).$$

2.2. **Additional properties of γ-factors.** From the axioms, one can deduce the following important property.

(viii) *(Local functional equation).* Let $(E/F, \pi, \psi) \in \mathscr{A}$. Then

$$\gamma(s, \pi, {}^{\otimes}\mathrm{I}, \psi)\gamma(1 - s, \tilde{\pi}, {}^{\otimes}\mathrm{I}, \overline{\psi}) = 1.$$

We let ν denote the character of $\mathrm{GL}_2(E)$ given by $|\det(\cdot)|$. We also have

(ix) *(Twists by unramified characters).* Let $(E/F, \pi, \psi) \in \mathscr{A}$. Then

$$\gamma(s, \pi \cdot \nu^{s_0}, {}^{\otimes}\mathrm{I}, \psi) = \gamma(s + s_0, \pi, {}^{\otimes}\mathrm{I}, \psi).$$

3. Main equality

Let $\mathscr{G}$ be the class of triples $(E/F, \sigma, \psi)$ consisting of: a degree 3 étale algebra E over a non-Archimedean local field F; an irreducible d dimensional Frob-semisimple representation σ of $\mathcal{W}'_E$; a non-trivial character $\psi : F \to \mathbb{C}^\times$.

If σ is a representation of $\mathcal{W}'_E$, of dimension d, we may form its tensor induction $\mathrm{I}^{\otimes}(\sigma)$, which is a representation of $\mathcal{W}'_F$, of dimension d^3 [5]. Let $\mathrm{Ver} : \mathcal{W}_F^{\mathrm{ab}} \to \mathcal{W}_E^{\mathrm{ab}}$ be the transfer [5]. If σ has dimension one, given by a character η of $\mathcal{W}_E^{\mathrm{ab}}$, then $\mathrm{I}^{\otimes}(\sigma)$ is given by the character $\eta \circ \mathrm{Ver}$ of $\mathcal{W}_F^{\mathrm{ab}}$. In a similar vein, if η is a character of $\mathcal{W}_E^{\mathrm{ab}}$, twisting σ by η gives $\mathrm{I}^{\otimes}(\sigma)$ twisted by $\eta \circ \mathrm{Ver}$. Recall that, if η corresponds via class field theory to a character χ of $E^\times$, then $\eta \circ \mathrm{Ver}$ corresponds to $\chi|_{F^\times}$. We fix $d = 2$.

Remark 3.1. As in the Asai representation in § 1, the construction is valid over a global field k and a separable cubic extension K/k. At a split place v of a global cubic extension K/k is $K_v \simeq k_v \times k_v \times k_v$, the cubic induction process simply consists in taking the tensor product $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$ of 2 dimensional representations of $\mathcal{W}'_F$.

3.1. **Main equality.** We consider triples $(E/F, \pi, \psi) \in \mathscr{A}$ and $(E/F, \sigma, \psi) \in \mathscr{G}$ such that $\pi \leftrightarrow \sigma$ correspond to each other via the local Langlands conjecture for GL_2 [17]. Additionally, we say that a triple $(E/F, \sigma, \psi) \in \mathscr{G}$ for a cubic extension of non-Archimedean fields is dihedral, tetrahedral or octahedral, depending on σ. We note that icosahedral Galois representations do not arise as smooth representations of GL_2.

Theorem 3.2. *Let $(E/F, \pi, \psi) \in \mathscr{A}$ and $(E/F, \sigma, \psi) \in \mathscr{G}$ be such that $\pi \leftrightarrow \sigma$ are related via the local Langlands correspondence. Then*

$$\gamma(s, \pi, \mathrm{I}^{\otimes}, \psi) = \gamma(s, \mathrm{I}^{\otimes}(\sigma), \psi).$$

Proof. As in [12, 13, 14, 8], we use a local-global approach which goes back to the early days of local class field theory. It is at the core of the Langlands-Shahidi method, and also in Deligne's construction of partial L-functions and γ-factors for representations of $\mathcal{W}'_F$.

Let $(E/F, \pi, \psi) \in \mathscr{A}$ and $(E/F, \sigma, \psi) \in \mathscr{G}$ be as in the statemen of the theorem. From § 2.1, Properties (iv) and (ix), we can assume π is supercuspidal unitary. Cuspidal representations of GL_2 are generic. Suppose we are given a global field K with a place v_0 such that $K_{v_0} \cong F$. We will concoct a globally generic cuspidal automorphic representation $\Pi = \otimes \Pi_v$ of $\mathrm{GL}_2(\mathbb{A}_K)$, with $\Pi_{v_0} \cong \pi$, where we excercise control over its ramification and its compatibility with the local Langlands conjecture. If K is a number field, we use Shahidi's γ-factors at Archimedean places, where we the main equality is known [24]. Let $\Psi = \otimes \Psi_v : K \backslash \mathbb{A}_K \to \mathbb{C}^\times$ be a non-trivial character, which we can assume satisfies $\Psi_{v_0} = \psi$, by Property (vi) of § 2.1. We have the global functional equation

$$L^S(s, \Pi, {}^{\otimes}\mathrm{I}) = \prod_{v \in S} \gamma(s, \Pi_v, {}^{\otimes}\mathrm{I}_v, \Psi_v) L^S(s, \widetilde{\Pi}, {}^{\otimes}\mathrm{I}). \tag{4}$$

Let us go through the argument in the case of $\mathrm{char}(F) = p$. The globalization is done via Theorem 3.3 of [13], being GL_2, the number of additional places where the representation is not necessarily unramified is one, which we call v_∞. We can also use the more general Theorem 1.1 of [8] for the globalization. At v_∞, we have that Π_{v_∞} is a tamely ramified principal series representation. We can apply multiplicativity, Property (iv), to get compatibility with the Galois side at v_∞. At every $v \notin \{v_0, v_\infty\}$, we have an unramified principal series representation, where the main equality is known. We then proceed as in the proof of Theorem 5.1 of [8] to obtain the main equality at the remaining place v_0. However, being GL_2, we have Drinfeld's proof of the global Langlands conjecture [7]. Thus, let Σ be the absolutely irreducible 2-dmensional λ-adic representation of the Weil group $\mathcal{W}_K$ corresponding to Π under Drinfeld's correspondence. Then we have the functional equation, proved by Deligne on the Galois side

$$L^S(s, {}^{\otimes}\mathrm{I}(\Sigma)) = \prod_{v \in S} \gamma(s, {}^{\otimes}\mathrm{I}_v(\Sigma_v), \Psi_v) L^S(s, {}^{\otimes}\mathrm{I}(\tilde{\Sigma})). \tag{5}$$

Now, if we compare equations (4) and (5), we obtain

$$\gamma(s, \Pi_v, {}^{\otimes}\mathrm{I}_{v_0}, \Psi_{v_0}) = \gamma(s, {}^{\otimes}\mathrm{I}_{v_0}(\Sigma_{v_0}), \Psi_{v_0}).$$

Which completes the proof of the main equality in characteristic p.

We now give more detail in the harder characteristic zero case. Assume thus F is an extension of $\mathbb{Q}_p$. If E is the separable algebra $F \times F \times F$, then from Property (v) of § 2.1 we have

$$\begin{aligned}\gamma(s, \pi, {}^{\otimes}\mathrm{I}, \psi) &= \gamma(s, \pi_1 \times \pi_2 \times \pi_3, \psi) \\ &= \gamma(s, \sigma_1 \otimes \sigma_2 \otimes \sigma_3, \psi).\end{aligned}$$

Each π_j is a smooth irreducible representation of $\mathrm{GL}_2(F)$ with corresponding degree 2 Weil-Deligne representaiton σ_j unde loal Langlands. We then also assume E/F is a cubic extension of p-adic fields.

First, consider the case σ is dihedral (that is, the image of σ in $\mathrm{PGL}_2(\mathbb{C})$ is a dihedral group); note that σ is always dihedral when p is odd. Then there is a quadratic extension E' of E and a character $\chi : E' \to \mathbb{C}^\times$, such that $\pi = \pi(\chi, \psi)$ is the Weil representation.

We lift F to a number field k, which is totally real and with only one place lying above 2 if $p = 2$; see for example [11]. We can also lift E/F to a cubic extension K/k, which is again totally real. We further lift E'/E to a quadratic extension K'/K. Let χ be the character of E' from which σ is induced. Then, we can always extend χ to a character $\mathcal{X}$ of $\mathcal{W}_{K'}$ which is unramified at all finite places except the place v_0 corresponding to E'. We thus have the representation

$$\Sigma = \mathrm{Ind}(\mathcal{X})$$

of the Weil group of K. The Galois L-function $L(s, {}^{\otimes}\mathrm{I}(\Sigma))$ has a functional equation due to Artin, and the decomposition of the global constants into a product of local constants was shown by Deligne.

Let $\Pi = \Pi(\Sigma)$ be the corresponding Weil representation, so that Π_v corresponds to Σ_v at all places. Then Π satisfies the functional equation, Property (vii) of § 2.1. We know the equality at all unramified places, in addition to real places, since we are in the totally real setting. By comparing both functional equations, we conclude the main equality at the remaining place, i.e.,

$$\gamma(s, \pi, \mathrm{I}^{\otimes}, \psi) = \gamma(s, {}^{\otimes}\mathrm{I}(\sigma), \psi).$$

We now assume σ is tetrahedral or octahedral, so that $p = 2$. Twisting by an unramified character, if necessary, we may assume that σ is a Galois local representation. We can find an extension of global fields K/k (with only one place v_0 over p) and a Galois representation Σ which gives σ at v_0. And, indeed has the same image as σ. More precisely, σ factorizes through a Galois group E'/E, and we find a global K'/K of the same degree, yielding E'/E at v_0. Hence, seeing σ as a representation of $\mathrm{Gal}(K'/K)$ we get Σ. Now the strong Artin conjecture is known for Σ, so let Π be the corresponding cuspidal representation, for which Π_{v_0} is π. We have again functional equations on both sides, with the real places giving triple products. At unramified places, there being no problem. And, notice that at finite places other than v_0, have odd residue characteristis, hence γ-factors agree at these places by the dihedral case discussed above. And, we only have the remaining one place v_0 above 2, where the representation can be tetrahedral or octahedral. By comparing both functional equations, we conclude the main equality at v_0.

□

4. Consequences

An important consequence of the main equality is that local factors become stable under highly ramified twists. Indeed, this follows by using Theorem 3.2 and applying the results of [6] to the Galois side of the equality.

Corollary 4.1 (Stability). *Let $(E/F, \pi_i, \psi) \in \mathscr{A}$, for $i = 1$ or 2, be such that their central characters satisfy $\omega_{\pi_1} = \omega_{\pi_2}$. Then, for a sufficiently ramified character η of $E^\times$:*

$$\gamma(s, \pi_1 \otimes \eta, r, \psi) = \gamma(s, \pi_2 \otimes \eta, r, \psi).$$

4.1. **Equality for L-functions and ε-factors.** In order to obtain the equality for the remaining local factors, we recall how to obtain L-funtions and ε-factors from γ-factors. We begin with the tempered L-function conjecture, now proved for all Langlands-Shahidi L-functions in [10].

(x) *(Tempered L-functions).* For $(E/F,\pi,\psi) \in \mathscr{A}$ tempered, let $P_\pi(t)$ be the unique polynomial with $P_\pi(0)=1$ and such that $P_\pi(q_F^{-s})$ is the numerator of $\gamma(s,\pi,{}^{\otimes}\mathrm{I},\psi)$. Then
$$L(s,\pi,{}^{\otimes}\mathrm{I}) = \frac{1}{P_\pi(q_F^{-s})}.$$
is holomorphic and non-zero for $\Re(s) > 0$.

Property (x) when combined with the Property (viii) of γ-factors, gives us the following well defined notion of ε-factors.

(xi) *(Tempered ε-factors).* Let $(E/F,\pi,\psi) \in \mathscr{A}$ be tempered, then
$$\varepsilon(s,\pi,{}^{\otimes}\mathrm{I},\psi) = \gamma(s,\pi,{}^{\otimes}\mathrm{I},\psi)\frac{L(s,\pi,{}^{\otimes}\mathrm{I})}{L(1-s,\tilde{\pi},{}^{\otimes}\mathrm{I})}$$
is a monomial in q_F^{-s}.

We also have

(xii) *(Twists by unramified characters).* Let $(E/F,\pi,\psi) \in \mathscr{A}$, then
$$L(s+s_0,\pi,{}^{\otimes}\mathrm{I}) = L(s,\pi\cdot\nu^{s_0},{}^{\otimes}\mathrm{I}),$$
$$\varepsilon(s+s_0,\pi,{}^{\otimes}\mathrm{I},\psi) = \varepsilon(s,\pi\cdot\nu^{s_0},{}^{\otimes}\mathrm{I},\psi).$$

Finally, we have multiplicativity of local factors

(xiii) *(Multiplicativity).* Let $(E/F,\pi,\psi) \in \mathscr{A}$ be such that
$$\pi \hookrightarrow \mathrm{Ind}(\chi_1 \otimes \chi_2),$$
the generic constituent obtained from parabolic induction with χ_i, $i=1,2$, characters of $E^\times$. Identify χ_i with a character of $\mathcal{W}_E$ via class field theory. Then
$$\begin{aligned} L(s,\pi,{}^{\otimes}\mathrm{I}) =& L(s,\chi_1|_{F^\times})L(s,\chi_2|_{F^\times}) \\ & L(s,\mathrm{I}_{E/F}(\chi_1\chi_2^{-1})\otimes\chi_2|_{F^\times})L(s,\mathrm{I}_{E/F}(\chi_2\chi_1^{-1})\otimes\chi_1|_{F^\times}), \end{aligned}$$
$$\begin{aligned} \varepsilon(s,\pi,{}^{\otimes}\mathrm{I},\psi) =& \varepsilon(s,\chi_1|_{F^\times},\psi)\gamma(s,\chi_2|_{F^\times},\psi) \\ & \varepsilon(s,\mathrm{I}_{E/F}(\chi_1\chi_2^{-1})\otimes\chi_2|_{F^\times},\psi)\varepsilon(s,\mathrm{I}_{E/F}(\chi_2\chi_1^{-1})\otimes\chi_1|_{F^\times},\psi), \end{aligned}$$
where $\mathrm{I}_{E/F}$ denotes induction from $\mathcal{W}_E$ to $\mathcal{W}_F$.

We deduce the main equality for L-functions and rood numbers from Theorem 3.2, with the aid of Langlands classification for GL(2) [15]. First for tempered representations, by Properties (x) and (xi), where we observe that all supercuspidal representations π of $\mathrm{GL}_2(E)$ are generic. Then in general, relying on Properties (xii) and (xiii).

Corollary 4.2. *Let $(E/F,\pi,\psi) \in \mathscr{A}$ and $(E/F,\sigma,\psi) \in \mathscr{G}$ be such that $\pi \leftrightarrow \sigma$ are related via the local Langlands correspondence. Then*
$$L(s,\pi,{}^{\otimes}\mathrm{I}) = L(s,{}^{\otimes}\mathrm{I}(\sigma)),$$
$$\varepsilon(s,\pi,{}^{\otimes}\mathrm{I},\psi) = \varepsilon(s,{}^{\otimes}\mathrm{I}(\sigma),\psi).$$

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