XLI SEMANA DE LA MATEMÁTICA
PONTIFICIA UNIVERSIDAD CATÓLICA DE VALPARAÍSO
Octubre 2015

CONFERENCIA

STONE'S DUALITY FOR BOOLEAN ALGEBRAS

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RESUMEN

The main aim of this talk is to present the famous Stone's representation theorem for Boolean algebra. We shall present all required definition in order to understand the construction, and the key argument of the proof.

A finite Boolean algebra is isomorphic to the Boolean algebra of subsets of a finite set. One can take, for example, the set of atoms of the Boolean algebra, and map an element to the set of atoms below this element.

This simple construction cannot be generalized in the infinite case. First appears a cardinality problem: there exists countable Boolean algebra, on the other hand a power set cannot be countable. Stone proved that each Boolean algebra can be embedded into a power set Boolean algebra, in a natural way.
The construction is more complicated. In general an element cannot be recognized only by the atoms below it. There even exist Boolean algebras with no atoms.

The idea of Stone's construction is to consider ultrafilters, as some sort of ``missing atoms´´. For a finite Boolean algebra, ultrafilters correspond to atoms, but in general there are much more. Moreover the set of ultrafilters can be endowed with a natural topology. This topology is compact, Hausdorff, and totally disconnected (such space are called Stone space).

Conversely, considering a Stone space, the set of closed-open set is a Boolean algebra for the usual set operations.

Stone proved that these constructions form a natural equivalence between the category of Boolean algebras and the (opposite) category of Stone spaces. In particular a Boolean algebra is isomorphic to the Boolean algebra of closed-open set of its topological space of ultrafilters. The isomorphism map an element to the set of all ultrafilter ``below´´ the element.