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Plenary talks

The Lorenz system near the loss of the foliation condition

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The well-known Lorenz system is classically studied via its reduction to the one-dimensional Lorenz map, which captures the full behaviour of the dynamics of the system [1]. The reduction requires the existence of a stable foliation. We study a parameter regime where this so-called foliation condition fails for the first time and, subsequently, the Lorenz map no longer accurately represents the dynamics. To this end, we study how the three-dimensional phase space is organised by the global invariant manifolds of saddle equilibria and saddle periodic orbits. Specifically, we explain and define two phenomena, observed by Sparrow in the 1980's [2]. First, the so-called flipping of the one-dimensional stable manifolds $W^s(p_\pm)$ of the secondary equilibria $p_\pm$ from one side to the other. Secondly, the development of hooks in the Poincaré return map that marks the loss of the foliation condition.

To investigate both these phenomena, we make extensive use of the continuation of orbit segments formulated by two-point boundary value problems [3]. We characterise geometrically a bifurcation in the α-limit of $W^s(p_\pm)$, which we call an α-flip. We accurately compute the parameter value at which this first α-flip occurs and find many subsequent α-flips. We then calculate the intersection curves of the two-dimensional unstable manifold $W^u(\Gamma)$ of a periodic orbit Γ with the classic Poincaré section. We identify and calculate when hooks form in the Poincaré map as a point of tangency of $W^u(\Gamma)$ with the stable foliation. We continue both the α-flip and tangency points in two parameters to a codimension-two bifurcation point, known as a T-point.

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Applications and asymptotic dynamics of piecewise contracting maps

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In this talk we are interested in the dynamical properties of piecewise contracting maps. We first show how this class of dynamical systems appears in the modeling of dissipative systems interacting in a nonlinear way, such as biological networks. Then, we are interested in the phenomenology of the asymptotic dynamics of these maps. For a wide class of such maps we give sufficient conditions to ensure some general basic properties, such as the periodicity, the total disconnectedness or the zero Lebesgue measure of the attractor. These conditions show in particular that a non-periodic attractor necessarily contains discontinuities of the map. Under this hypothesis, we obtain numerous examples of attractors, ranging from finite to connected and chaotic, contrasting with the (quasi-)periodic asymptotic behaviors observed so far.

Interacting invariant sets in a 2D noninvertible map model of wild chaos

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We consider a two-dimensional noninvertible map that was introduced by Bamón, Kiwi and Rivera-Letelier in 2006 as a model of wild Lorenz-like chaos [1,2]. The map acts on the plane by opening up the critical point to a disk and wrapping the plane twice around it; points inside the disk have no preimage. The bounding critical circle and its images, together with the critical point and its preimages, form the critical set. This set interacts with a saddle fixed point and its stable and unstable sets. Advanced numerical techniques enable us to study how the stable and unstable sets change as a parameter is varied along a path towards the wild chaotic regime. We find four types of bifurcations: the stable and unstable sets interact with each other in homoclinic tangencies (which also occur in invertible maps), and they interact with the critical set in three types of tangency bifurcations specific to this type of noninvertible map; all tangency bifurcations cause changes to the topology of these global invariant sets. Overall, a consistent sequence of all four bifurcations emerges, which we present as a first attempt towards explaining the geometric nature of wild chaos. Using two-parameter bifurcation diagrams, we show that essentially the same sequences of bifurcations occur along different paths towards the wild chaotic regime, and we use this information to obtain an indication of the size of the parameter region where wild Lorenz-like chaos is conjectured to exist. We further continue these bifurcations into a regime that corresponds to a contracting Lorenz-like attractor, where we find regions of wild dynamics, as well as multistability and chaotic transients.

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Global manifolds and the transition to chaos in the Lorenz system

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The Lorenz system still fascinates many people because of the simplicity of the equations that generate such complicated dynamics on the famous butterfly attractor. This talk addresses the role of the stable and unstable manifolds in organising the dynamics more globally. A main object of interest is the stable manifold of the origin of the Lorenz system, also known as the Lorenz manifold. This two-dimensional manifold and associated manifolds of saddle periodic orbits can be computed accurately with numerical methods based on the continuation of orbit segments, defined as solutions of suitable two-point boundary value problems [3]. We use these techniques to give a precise geometrical and topological characterisation of global manifolds during the transition from simple dynamics, via preturbulence to chaotic dynamics, as the Rayleigh parameter of the Lorenz system is increased [1, 2].

This is joint work with: Hinke Osinga (The University of Auckland) and Eusebius Doedel (Concordia University, Montreal)

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Interactions of forward- and backward-time isochrons

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In the 1970s Winfree [1] introduced the concept of an isochron as the set of all points in the basin of an attracting periodic orbit that converge to the periodic orbit in forward time with the same asymptotic phase. It has been observed that in slow-fast systems, such as the FitzHugh-Nagumo model [2], the isochrons of such systems can have complicated geometric features [3,4]; in particular, regions with high curvature that are related to sensitivity in the system. In order to understand where these features come from, we introduce backward-time isochrons that exist in the basin of a repelling periodic orbit, and we consider their interactions with the forward-time isochrons. We show that a cubic tangency between the two sets of isochrons is responsible for creating the high curvature features. We present two normal-form-type models that feature a cubic tangency bifurcation between forward- and backward-time isochrons, generated via two different mechanisms: the introduction of a global time-scale separation, and a local perturbation to the velocity along trajectories. This study makes use of a boundary value problem formulation to compute isochrons accurately as parametrised curves.

This is joint work with: Bernd Krauskopf and Hinke Osinga (The University of Auckland).

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Approximation of invariant measures: Ulam's method and beyond

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In 1960 Ulam proposed discretising the Perron-Frobenius operator for a non-singular map $(T; X)$ by projecting $L^1(X)$ onto the subspace of piecewise constant functions with respect to a fixed partition of subsets of X. Ulam's conjecture was that as the partition is refined, the fixed points of the approximation scheme should converge in L^1 to a fixed point of the Frobenius-Perron operator. Thus "Ulam's method" was born! Li (1976) proved the conjecture for piecewise C^2 expanding interval maps, and further results have been obtained by many authors over the subsequent decades. It is now clear that most of these results rely on strong analytical control of the spectrum of the Frobenius-Perron operator on suitable Banach spaces embedded in L^1. Indeed, in such settings, useful convergence rates can be obtained (for example, by using the spectral perturbation machinery of Keller and Liverani). However, applying these results to new classes of maps can be extremely difficult (or impossible); this is especially true examples coming from real applications. In this sense, a satisfactory proof of Ulam's conjecture remains elusive.

This talk will survey the ideas above, and describe recent progress within a variational framework in which Ulam's method arises as one possible approximation scheme (joint work with C Bose). Analytical proofs of convergence can come "cheaply" in settings where the spectral perturbation approach does not apply. Progress towards Ulam's conjecture and alternatives will be discussed.

The Ghys-Sullivan and Hector conjectures

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We will explain the main points in the solution to these longstanding conjectures in the real-analytic case: exceptional minimal sets have zero Lebesgue measure, and their complement is made of finitely many orbits of intervals.

This is joint work with B. Deroin and V. Kleptsyn.

Computing global invariant manifolds: Techniques and applications

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Global invariant manifolds play an important role in organising the behaviour of a dynamical system. Together with equilibria and periodic orbits, they form the so-called skeleton of the dynamics and offer geometric insight into how observed behaviour arises. In most cases, it is impossible to find invariant manifolds explicitly and numerical methods must be used to find accurate approximations. Developing such computational techniques is a challenge on its own and, to this date, the focus has primarily been on computing two-dimensional manifolds. Nevertheless, these computational efforts offer new insights that go far beyond a confirmation of the known theory. Furthermore, global invariant manifolds in dynamical systems theory not only explain asymptotic behaviour, but more recent developments show that they are equally useful for explaining short-term transient dynamics. This paper presents an overview of these more recent developments, in terms of novel computational methods, as well as applications that have stimulated recent advances in the field and highlighted the need for new mathematical theory.

This is joint work with: Bernd Krauskopf (The University of Auckland) and Eusebius Doedel (Concordia University, Montreal)

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Thermodynamic formalism of one-dimensional dynamical systems

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In their pioneer works, Sinai, Bowen, and Ruelle gave a complete description of the thermodynamic formalism of uniformly hyperbolic diffeomorphisms and Hölder continuous potentials. In this talk, I'll report on recent progress in real and complex dimension 1, where a complete picture is emerging. For simplicity the talk will be restricted to geometric potentials and the quadratic family, but most results apply in greater generality. The first goal is to describe the (non-)existence of equilibrium states, their statistical properties, and the real analytic properties of the geometric pressure function. The second goal is to describe phase transitions: The phenomenon of lack of real analyticity. After classifying and describing the mechanisms that produce phase transitions, the focus will be on the various surprising phenomena that occur at criticality, some of which illustrate the universality of the quadratic family.

Existence of centrally contractive Lorenz attractors

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In this talk we explore the existence of centrally contractive Lorenz attractors in dimension three. We start recalling the Lorenz equations, for which numerical experiments performed by Lorenz around the mid-sixties suggested the existence, for some real parameters of a strange attractor: a set which traps the positive orbit of all points in an open neighborhood of it, displays sensitive dependence to initial conditions and, in some way, it seems robust. Moreover, it is not hyperbolic because of the robust accumulation of singularities by non-wandering regular orbits of the flow. In view of the non-existence of an explicit solution for the Lorenz system, Guckenheimer's geometric model, also known as the *Geometric Lorenz Attractor*, has been introduced. We will review the geometric model and its essential properties as well as the centrally contractive model. We will present some results showing that this type of attractors appears via the unfolding of singular cycles.

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Stability of complex Hénon maps

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In the study of complex dynamics in several variables, the study of complex Hénon maps in $\mathbb{C}^2$ is a first step for the global understanding of holomorphic dynamics in higher dimensions.

In this talk we will present several notions "near" hyperbolicity and study these phenomena when the system is partially hyperbolic in the Julia set J, which is equivalent in this context to dominated splitting in J.

Student talks

Chaos in a mathematical model of a cold-sensitive nerve-ending

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Neurons respond to external stimuli with electrical pulses called action potentials or spikes. Some neurons have spontaneous activity in a variety of firing patterns. We are characterizing a model inspired in cold thermoreceptors, which responds with different patterns of spike trains depending on the input parameter temperature [1]. With the help of methods of the theory of dynamical systems we are analyzing the behavior of the model and its sensitivity to changes in parameters. We have found that in this model [1] chaos arises naturally for several ranges of parameters. This is relevant because the model, although inspired in cold thermoreceptors, presents characteristics that are common to many neurons in the CNS. Future work will concentrate in the consequences of the chaotic behavior on the neural coding.

This is joint work with P. Orio (CINV).

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Functional norms for generalized bakers transformations

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Generalized Bakers Transformations (GBTs) are Lebesgue measure preserving maps of the unit square. Mixing properties of a non-singular map can be analyzed via a transfer operator which acts on an appropriate Banach space of probability densities. When a non-singular map has some hyperbolicity there are several common transfer operator methods. One is to study the transfer operator associated to an expanding factor. Another method, which we will explore in this talk, is to construct Banach spaces of densities for the full hyperbolic map with anisotropic norms. We will apply this paradigm to the transfer operators of certain piecewise smooth uniformly hyperbolic GBTs and illustrate how the method could be used effectively for non-uniformly hyperbolic examples.

Global invariant manifolds near a homoclinic flip bifurcation

Andrus Giraldo
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Homoclinic bifurcations, such as the well-known Shilnikov bifurcation, may give rise to complicated dynamics including oscillations and chaos. Our work is on the homoclinic flip bifurcation, which is a codimension-two bifurcation. Its characterising feature is that the associated stable (or unstable) manifolds change from being orientable to non-orientable. The homoclinic flip bifurcation gives rise to additional, different bifurcation curves. We are interested in understanding how the associated global invariant manifolds interact and what this means for the overall dynamics when these curves are crossed.

This talk will focus on explaining the unfolding of the homoclinic flip bifurcation in a particular case that has not been studied before. Here, there are bifurcating saddle periodic orbits, with invariant manifolds that can also be orientable or non-orientable. In our study the invariant manifolds are computed via the continuation of solutions to a two-point boundary value problem.

This is joint work with: Bernd Krauskopf and Hinke Osinga (The University of Auckland).

On the dynamics of impulsive semiflows

Nelda Jaque
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In this talk we will introduce the concept of impulsive dynamical systems. We are interested in the study of ergodic and topological properties that remain valid on the setting of impulsive semiflows. As an example we will present a result of Alves and Carvalho showing the existence of invariant probability measures.

Invariant and slow manifolds in the singular Hopf bifurcation

José Mujica
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Mixed-mode oscillations are orbits of a vector field with oscillations featuring an alternation of both small and large amplitudes. They appear in the context of multiple-time-scales systems and may be organised by different structures, including a singular Hopf bifurcation. At this bifurcation, in a system with one fast and two slow variables, an equilibrium crosses a fold curve of the so-called critical manifold of the reduced system. As a consequence, in the full system the equilibrium undergoes a Hopf bifurcation extremely close to the fold curve. Desroches et al. [1] studied a normal form of the singular Hopf bifurcation due to Guckenheimer [2] and detected a tangency between a repelling slow manifold and the unstable manifold of the saddle-focus, this phenomenon is associated with mixed-mode oscillations. Motivated by this work we employ a boundary value problem setup to compute the relevant surfaces as families of orbit segments. This allows us to study how their interaction shapes the phase space locally and globally.

This is joint work with: Bernd Krauskopf and Hinke Osinga (The University of Auckland).

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Lyapunov exponents

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Lyapunov exponents are used in different fields of mathematics and it has an significant role in differentiable ergodic theory.

We shall give a general vision of Lyapunov exponents. If (X, μ) is a probability space and $f : X \longrightarrow X$ is an invertible map preserving μ, Oseledec's Multiplicative Ergodic Theorem [3] says that the set of regular point is f-invariant and with full measure.

We find some relations between Lyapunov exponents and entropy (Ruelle inequality [4], Pesin's entropy formula [2]) and when this relation guarantees the existence of non-zero exponents.

Finally, we show the connection between Lyapunov exponents and hyperbolic sys- tems, and non uniformly hyperbolic systems.

If we have more time, I would like to talk about some more propierties of hyperbolic measures (all the exponents are non-zero), en particular, show the Ledrappier-Young's entropy formula [1].

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